

SHORT-TIME CRITICAL DYNAMICS

L. SCHÜLKE

Fachbereich Physik, Universität Siegen, D-57068 Siegen, Germany

E-mail: schuelke@pollux.physik.uni-siegen.de

An introductory review to short-time critical dynamics is given. From the scaling relation valid already in the early stage of the evolution of a system at or near the critical point, one derives power law behaviour for various quantities. By a numerical simulation of the system one can measure the critical exponents and, by searching for the best power law behaviour, one can determine the critical point. Critical slowing down as well as finite size corrections are nearly absent, since the correlation length is still small for times far before equilibrium is reached. By measuring the (pseudo) critical points it is also possible to distinguish (weak) first-order from second-order phase transitions.

In this talk I like to give an introductory report about the main features of short-time critical dynamics. The topic exists since about ten years ago. It will be shown that it is possible to measure the critical temperature as well as all the static and dynamic critical exponents already in the short-time regime of the evolution of statistical systems, far before the equilibrium is reached. This can be done by starting either from an initial system of very high temperature, suddenly quenched to the critical temperature, or by starting with a completely ordered state, leading to the same results. For a comprehensive review we refer to references¹ and². It will also be shown that short-time critical dynamics can in addition be used as a tool to distinguish first-order from second order phase transitions by comparing the critical temperature obtained from the different starting conditions.

It has for long been known that statistical system for particular values of their coupling show critical behaviour. This behaviour is mainly characterised by the fact that the correlation length ξ_x becomes infinite as well as the corresponding correlation time ξ_t . If we define the difference between the coupling and the critical one by τ , we have

$$\xi_x \rightarrow \tau^{-\nu}, \quad \xi_t \rightarrow \tau^{-\nu z}; \quad \tau = \frac{K - K_c}{K_c}. \quad (1)$$

Other quantities approach zero or infinity when $\tau \rightarrow 0$, e.g., the magnetisation of a spin system behaves as

$$M(\tau) \rightarrow (\tau)^\beta. \quad (2)$$

The exponents ν , β and z are static resp. dynamic critical exponents. Mainly due to the large correlation length and -time there exists a dynamical scaling form. We do not explicitly quote the scaling form here, but refer to it in the subsequent discussion.

In 1989 Janssen, Schaub and Schmittmann³ have shown, that this scenario is not only valid in equilibrium, but *a scaling relation is already valid in the short-time regime* of the evaluation of a critical system. The system should be prepared at a very high temperature, with a small magnetisation m_0 remaining. It is then suddenly quenched to the critical temperature and released to the dynamical evolution of model A. The authors show, using renormalisation group analysis, that after a macroscopic small time t_{mic} a scaling form is valid. For the k -th moment of the magnetisation it reads

$$M^{(k)}(t, \tau, L, m_0) = b^{-k\beta/\nu} M^{(k)}(t/b^z, b^{1/\nu}\tau, L/b, b^{x_0}m_0). \quad (3)$$

In this equation b is an arbitrary scaling factor, t is the time, τ is defined in Eq.(1), L is the linear lattice size, and $m_0 \gtrsim 0$ the initial magnetisation. Except of this additional last argument the scaling form looks exactly like that in equilibrium. This argument m_0 gives rise to a new, independent critical exponent x_0 , the scaling dimension of the initial magnetisation.

The questions arises how to exploit the scaling relation (3) in order to get the desired information about the critical exponents. We investigate the system numerically with Monte-Carlo simulation at a temperature at or near the critical temperature T_c , measuring the magnetisation and its second moment, $M(t)$ and $M^{(2)}(t)$, as well as the autocorrelation $A(t) = \langle \sum_i S_i(t) S_i(0) \rangle$, where a time unit is defined as a complete update of the whole lattice. Since the system is in the early stage of the evolution the correlation length is still small and finite size problems are nearly absent. Therefore we generally consider L large enough and skip this argument. An occasional check is made to prove this assumption. We now choose the scaling factor $b = t^{1/z}$ so that the main t -dependence on the right is cancelled. Expanding the scaling form (3) for $k = 1$ with respect to the small quantity $t^{x_0/z} m_0$, one obtains

$$M(t) \sim m_0 t^\theta F(t^{1/\nu z} \tau) = m_0 t^\theta \left(1 + at^{1/\nu z} \tau + O(\tau^2) \right), \quad (4)$$

where $\theta = (x_0 - \beta/\nu)/z$ has been introduced. In most cases $\theta > 0$, i.e., for $\tau = 0$ the small initial magnetisation *increases* in the short-time region. Figure 1 shows the initial increase for the 3-dimensional Ising model as an example⁴. For small $\tau \neq 0$ we have expanded the function $F(t^{1/\nu z} \tau)$. There appear corrections to the simple power law dependent on the sign of τ . Therefore

simulating the system for values of the coupling K in the neighbourhood of the critical point one obtains $M_K(t)$ with non-perfect power law behaviour, and the critical point K_c can be obtained by interpolation. Figure 2 shows a plot of the magnetisation for the 2-dimensional 3-state Potts model for three different values of the coupling⁵. The curve with the best power law behaviour is found for $K_c = 1.0055(8)$, while the exact value is 1.00505.

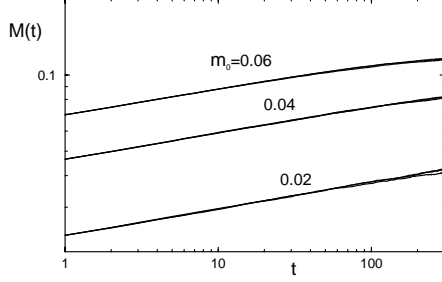


Figure 1. Three-dimensional Ising model: Initial increase of the magnetisation for different initial magnetisations m_0 , plotted vs. time on log-log scale. $L = 128$. Result is $\theta = 0.108(2)$

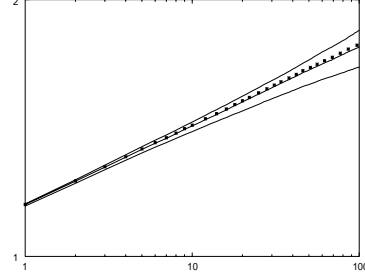


Figure 2. Two-dimensional 3-state Potts model: Magnetisation for three different values of the coupling near the critical point. From above $J = 1.00750, 1.00505$, and 1.0025 . Dotted curve $J_c = 1.0055(8)$

For the second moment of the magnetisation one can deduce $M^{(2)}(t) \sim L^{-d}$ ($d = \text{dimension of the system}$), because the correlation length is still small in the early time region even at the critical point. Combining this with the result of the scaling form for $\tau = 0$ and $b = t^{1/z}$, $M^{(2)}(t) \sim t^{-2\beta/\nu z} M^{(2)}(1, t^{-1/z} L)$, one obtains the power law

$$M^{(2)}(t) \sim t^{c_2} \quad c_2 = \left(d - 2\frac{\beta}{\nu}\right) \frac{1}{z}. \quad (5)$$

An example for the power law behaviour of this quantity is given⁴ in Fig. 3.

From the scaling form (3), setting again $b = t^{1/z}$, one derives for $\partial_\tau \ln M(t, \tau)|_{\tau=0}$ the power law

$$\partial_\tau \ln M(t, \tau)|_{\tau=0} \sim t^{\frac{1}{\nu z}}. \quad (6)$$

Approximating numerically the derivative of $M(t)$ involves the difference of small quantities and is therefore affected more by uncertainties. However, Fig. 4 shows for the 2-dimensional 3-state Potts model as example that the data here also, for $t > 20$, show perfect power law behaviour.

Another interesting quantity is the autocorrelation

$$A(t) = \frac{1}{L^d} \left\langle \sum_i S_i(t) S_i(0) \right\rangle. \quad (7)$$

An analysis⁶ (for $m_0 = 0$) leads to

$$A(t) \sim t^{c_a} \quad c_a = \frac{d}{z} - \theta. \quad (8)$$

The plot in Fig. 5 shows a nearly perfect power law. Again the 3-dimensional Ising model has been taken as an example⁴.

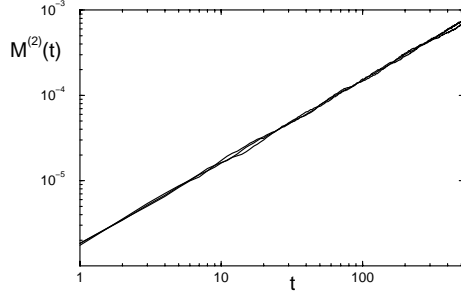


Figure 3. Three-dimensional Ising model: Second moment of the magnetisation, plotted vs. time on log-log scale. $L = 128$. From the slope one gets $c_2 = 0.970(11)$

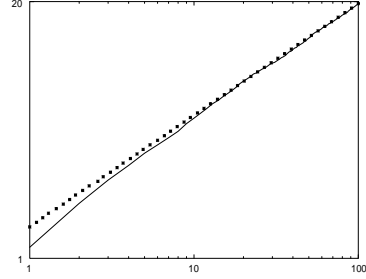


Figure 4. Two-dimensional 3-state Potts model: $\partial_\tau \ln M(t, \tau)$, plotted vs. time on log-log scale. Dots represent the straight line fitted to the curve within the time interval $[20 \dots 100]$

Care has to be taken to exclude the time region $t < t_{mic}$, as well as the large time region where deviations of the the power law behaviour occur due to finite size effects or due to too large fluctuations. An example for a more detailed analysis of the straight line (on log-log plot) in Fig. 5 is given in Fig. 6. The individual slope in bins between t and $1.5 t$ is plotted versus t . It is seen that the slope for $t < \sim 10$ is not yet constant and therefore the region $t < 10$ should be excluded. It is also seen that the fluctuations within the higher bins increase although much more points enter.

Till now a completely disordered initial state has been considered as starting point, i.e., a state of very high temperature. The question arises how a *completely ordered initial state* evolves, when heated up suddenly to the critical temperature. In the scaling form (3) one can skip besides L , also the

argument $m_0 = 1$:

$$M^{(k)}(t, \tau) = b^{-k\beta/\nu} M^{(k)}(t/b^z, b^{1/\nu}\tau) \quad (m_0 = 1). \quad (9)$$

The system is simulated numerically by starting with a completely ordered state, whose evaluation is measured at or near the critical temperature. The quantities measured are $M(t)$ and $M^{(2)}(t)$. With $b = t^{1/z}$ one avoids the main t -dependence on the right of eq.(9), and for $k = 1$ one has

$$M(t, \tau) = t^{-\beta/\nu z} M(1, t^{1/\nu z} \tau) = t^{-\beta/\nu z} \left(1 + a t^{1/\nu z} \tau + O(\tau^2)\right). \quad (10)$$

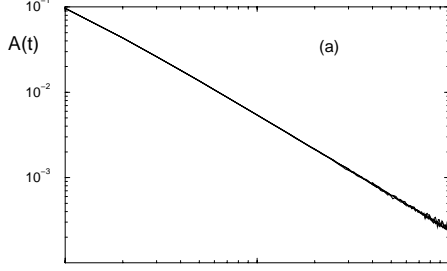


Figure 5. Three-dimensional Ising model: Autocorrelation plotted vs. time on log-log scale. $L = 128$. From the slope one gets $c_a = 1.36(1)$

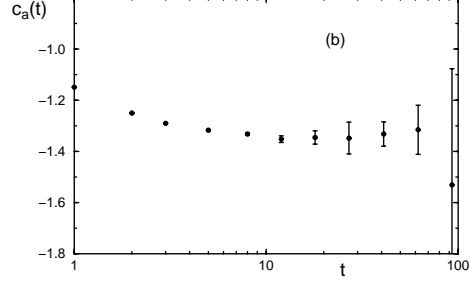


Figure 6. Three-dimensional Ising model: slope c_a measured between t and $1.5t$ plotted vs. time on log-log scale.

Similarly as in case of the a completely disordered start, see eq.(4), one can choose couplings in the neighbourhood of the critical point and fix from these measurements the critical point by looking for the best power law behaviour. We would like to mention that measurements starting from $m_0 = 1$ are much less affected by fluctuations, because the quantities measured are rather big in contrast to those from a random start. Figure 7 is an example of the 3-dimensional Ising model⁴. Both the critical point as well as the slope at that point can be determined from the measurement. Figure 8 shows the decay of the magnetisation at the critical point for different lattice sizes from $L = 32$ to 256. The data points for all lattice sizes agree completely up to $t = 1000$, only the data for $L = 32$ show some deviation for $t_{>} \sim 300$. This justifies our statement that finite size effects are unimportant in short-time critical dynamics.

Also for the cold start one derives from (9)

$$\partial_\tau \ln M(t, \tau)|_{\tau=0} \sim t^{1/\nu z} \quad (11)$$

which allows to measure the ratio $1/\nu z$. With the magnetisation and its second moment the time dependent Binder cumulant

$$U(t) = \frac{M^{(2)}}{(M)^2} - 1 \sim t^{d/z} \quad (12)$$

is defined. From its slope one can directly measure the dynamic exponent z . For the last two quantities we give examples in Figs. 9 and 10, using the 3-dimensional Ising model.

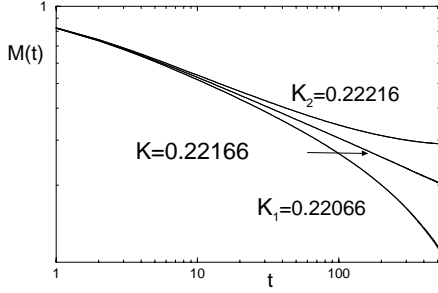


Figure 7. Three-dimensional Ising model: Decay of the magnetisation for initial magnetisations $m_0 = 1$, for three different values of the coupling near the critical point.

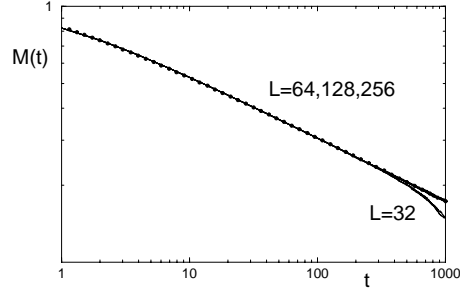


Figure 8. Three-dimensional Ising model: Decay of the magnetisation for different lattice sizes at the critical point.

In summarising the topics discussed up to now, we would like to make the following remarks:

- From an investigation of the system from a high-temperature initial state, a new independent critical exponent θ can be determined.
- A determination of the critical point and of all the critical exponents is possible starting from a high-temperature or from a zero-temperature initial state. The results are the same for second-order transitions.
- In contrast to investigations in equilibrium, the correlation length is still finite in the short-time regime, therefore finite size effects are strongly reduced.
- Also the correlation time is still finite, therefore critical slowing down is absent.
- We work directly with sample averages, not with time averages.

As is well known, critical slowing down can be overcome in some cases by the successful non-local cluster algorithm⁷. However, the cluster algorithm is not applicable, e.g., for systems with randomness or frustration. With short-time dynamic simulations we have, e.g., successfully investigated the chiral degree of freedom in the 2-dimensional fully frustrated XY model, where critical slowing down makes measurements in equilibrium extremely difficult^{8,9}.

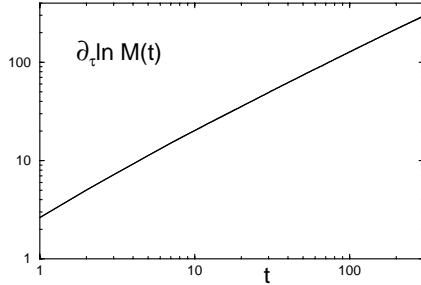


Figure 9. Three-dimensional Ising model: Logarithmic derivative of the magnetisation with respect to τ , obtained from an ordered start.

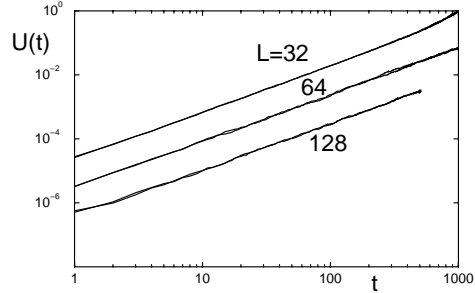


Figure 10. Three-dimensional Ising model: Time evolution of the Binder cumulant $U(t)$ for different lattice sizes.

Till now we have investigated critical systems which undergo a second order phase transition. The relaxation of initial states with very high temperature suddenly quenched to the critical temperature, or of initial states with zero temperature, suddenly heated up to the critical temperature, is measured in the short-time regime. Various quantities exhibit a power law behaviour. From measurements with couplings in the neighbourhood of the critical point we can find by interpolation the optimal power law behaviour and determine in this way the critical point. The result is the same for both initial conditions within the statistical error.

If the phase transition is first order, at least if it is weak, the behaviour should be similar to second order. By cooling down a very hot initial state, one should find a *pseudo critical point* K^* , and by heating up a cold initial state *another pseudo critical point* K^{**} is found. We expect

$$K^{**} < K_c < K^*.$$

A difference between K^{**} and K^* is a clear signal for a first-order transition.

The p -state Potts model undergoes a second order phase transition for $p \leq 4$ at the critical point $K_c = \ln(1 + \sqrt{p})$. For $p \geq 5$ the transition is of

first order, but still weak for small p . We have investigated this model¹⁰ for $p = 5$ and $p = 7$. For the initial condition $m_0 = 0$ the second moment $M^{(2)}(t)$ of the magnetisation has been measured, and for initial condition $m_0 = 1$ the quantities $M(t)$ and $M^{(2)}(t)$. In case of $p = 7$ the time interval extended up to $t = 6\,000$ for $L = 280$ and 560 , while in case of $p = 5$ a more careful analysis is necessary. Here we have chosen time intervals up to $40\,000$ for $L = 560$. In both cases we find a clear difference between K^* and K^{**} . In case of K^{**} , both the magnetisation and the Binder cumulant (12) have been used. Figure 11 shows the result for $p = 7$. The results for $p = 5$ are similar. This example shows that short-time critical dynamics can be successfully used as a tool to distinguish between first- and second order phase transitions.

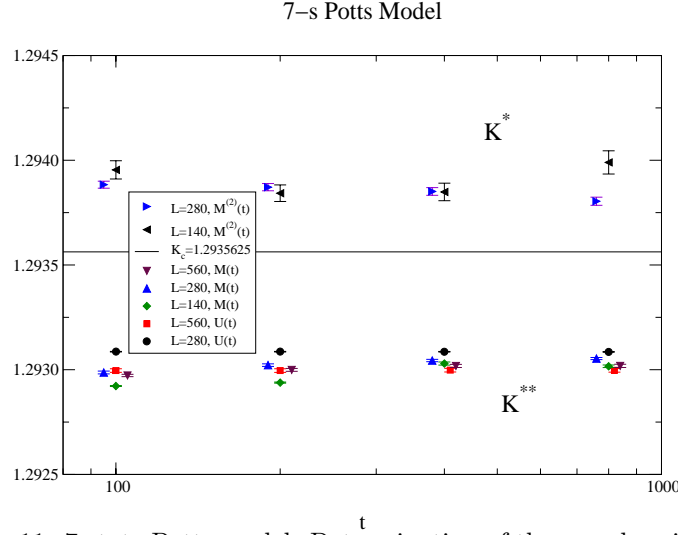


Figure 11. 7-state Potts model: Determination of the pseudo-critical point K^* determined from $M^{(2)}(t)$ and of K^{**} determined from $M(t)$ and $U(t)$. For the time interval $[t \cdots t_{max}]$ used for the fit various starting points between $t = 100$ and 800 and the maximum time have been used.

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